

## **A Perspective on Reliability: Probability Theory and Beyond**

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### **INTRODUCTION**

Reliability assessment in the coming era is inclined to be characterized by a difficult dilemma. On the one hand units and systems will be required to be ultra reliable; on the other hand, it may not be possible to subject them to a full-scale testing. A case in point occurs where testing is limited is one-of-a-kind complex systems, such as space exploration vehicles or where severe testing constraints are imposed such as full scale testing of strategic nuclear weapons prohibited by test ban treaties and international agreements. Decision makers also require reliability assessments for problems with terabytes of data, such as from complex simulations of system performance. Quantitative measures of reliability and their associated uncertainties will remain integral to system monitoring and tactical decision making. The challenge is to derive these defensible measures in light of these dilemmas.

Because reliability is usually defined as a probability that the system performs to its required specification, probability enters into the heart of these dilemmas, both philosophically and practically. This paper provides an overview of the several interpretations of probability as they relate to reliability and to the uncertainties involved. The philosophical issues pertain to the interpretation and the quantification of reliability. For example, how must we interpret a number like  $10^{-9}$ , for the failure rate of an airplane flight or an electrical power plant? Such numbers are common, particularly in the context of safety. Does it mean one failure in  $10^9$  identical, or almost identical, trials? Are identical trials physically possible, let alone the fact that  $10^9$  trials

can take generations to perform? How can we make precise the notion of almost identical trials? If the trials are truly identical, then all of them must produce the same outcome and so the reliability must be either one or zero. However tautologies, like certainty and impossibility, can be claimed only on the basis of pure logic, not empirical evidence. Thus, from a practical point of view, an interpretation of reliability that is devoid of abstractions like  $10^9$  identical trials, appears to be in order. It is the meaning of probability that lies at the core of the interpretation of reliability.

## WHAT IS PROBABILITY

To begin, Webster defines the mathematical meaning as *likelihood of the occurrence of any event in the doctrine of chances, or the ratio of the number of favorable chances to the whole number of chances, favorable and unfavorable*. Some famous quotations about probability include: “Probability is the appearance of the agreement or disagreement of two ideas, by the intervention of proofs whose connection is not constant, but appears for the most part to be so.” Locke. “The whole life of man is a perpetual comparison of evidence and balancing of probabilities.” Buckminster. “We do not call for evidence till antecedent probabilities fail.” J. H. Newman.

Theoretical definitions of probability encompass sets and fields resulting in a mathematical construct such as:

A set function  $P$  defined for all sets in a Boolean field  $F$  having these properties is referred to as the probability measure on  $F$ :

- For every event,  $E$ , in Boolean field,  $F$ , there is associated a real non-negative number  $P(E)$ , called the probability of event  $E$ .
- If  $E_1, E_2, \dots$  is a countably infinite sequence of mutually disjoint sets in  $F$  whose union is in  $F$  then  

$$P(\bigcup E_i) = \sum P(E_i)$$
- $P(R)=1$  ( $R$  is the sample space.)

$P$  is the probability measure (or probability distribution) on the Borel field  $F \rightarrow B(F)$  (Wilks, 1962).

Some notation is necessary to reformulate the above definition, hopefully, in more meaningful terms and relating it to reliability. Let  $E, E_1, \dots, E_i, \dots$ , denote several uncertain events of interest at some reference time, say  $\tau$ . It is common to set  $\tau=0$ ; however, it is important not to lose track of its presence. As an example,  $E$  could denote the event that a deployed system accomplishes its mission. The complement of  $E$  is denoted by  $\bar{E}$ , the event that a deployed system fails to accomplish its mission. Another example could be that  $E_i = \{T_i \geq t\}$ , where  $T_i$  denotes the life-time of the  $i$ -th sub-system of a deployed system, measured from the time of deployment of the system, and  $\{T_i \geq t\}$  denotes the event that the  $i$ -th sub-system functions for at least  $t$  units of time. Here,  $t$  is called the *mission time*.

Let  $\mathcal{H}$  denote the *history* or the *background information* that is available to the individual(s) contemplating the uncertain events, at time  $\tau$ . In principle,  $\mathcal{H}$  should encompass all that is known at time  $\tau$ ; scientific knowledge, engineering information, informed testimonies, design specifications, physical models, computer codes, judgement, preferences, and hard historical data on replicates of the uncertain event, if available. Thus at any time  $\tau$ , there is the known  $\mathcal{H}$ , and the unknown  $E, E_1, \dots, E_i, \dots$ .

The fundamental problem of the treatment of uncertainty is how the uncertainty about  $E, E_1, \dots, E_i, \dots$ , at the  $\tau$ , should be quantified in the light of  $\mathcal{H}$ .

The probability of an event, say  $E$ , in the light of  $\mathcal{H}$  at time  $\tau$ , is a number denoted by  $\mathcal{P}^\tau(E; \mathcal{H})$ , that is required to satisfy certain rules (or axioms) called the *calculus of probability*.

When the event  $E$  pertains to an ability to perform a certain function, an ability to survive a specified mission time, or the ability to produce a specified level of output, then  $\mathcal{P}^\tau(E; \mathcal{H})$  is known as the *reliability* of the item or unit which is required to function or to survive. Thus reliability is de facto the probability of a certain type of an event. When the item in question is a human subject, the term *survival analysis*, rather than reliability, is commonly used. As indicated above, the *mission time* need not be measured in the units of time; it could be other performance metrics such as miles traveled, rounds fired, cycles completed, or output produced.

The calculus of probability comprises of the following three rules, *convexity*, *addition*, and *multiplication*; these are given below in order:

- i)  $0 \leq \mathcal{P}^\tau(E; \mathcal{H}) \leq 1$ , for any event  $E$ ;
- ii)  $\mathcal{P}^\tau(E_1, \text{ or } E_2; \mathcal{H}) = \mathcal{P}^\tau(E_1; \mathcal{H}) + \mathcal{P}^\tau(E_2; \mathcal{H})$  for any two events  $E_1$  and  $E_2$  that are mutually exclusive—that is, they cannot simultaneously occur, and
- iii)  $\mathcal{P}^\tau(E_1 \text{ and } E_2; \mathcal{H}) = \mathcal{P}^\tau(E_1 | E_2; \mathcal{H}) \cdot \mathcal{P}^\tau(E_2; \mathcal{H})$  where  $\mathcal{P}^\tau(E_1 | E_2; \mathcal{H})$  is a quantification via probability of the uncertainty about an event  $E$ , supposing that event  $E_2$  has occurred.

The quantity  $\mathcal{P}^\tau(E_1 | E_2; \mathcal{H})$  is known as the *conditional probability* of  $E_1$ , given  $E_2$ . It is important to note that conditional probabilities are in the subjunctive. That is, the disposition of  $E_2$  at time  $\tau$ , were it to be known, would become a part of the history  $\mathcal{H}$  at time  $\tau$ . The vertical line between  $E_1$  and  $E_2$  represents a supposition or assumption about the occurrence of  $E_2$ . Finally,  $\mathcal{P}^\tau(E_1 \text{ and } E_2; \mathcal{H})$  can also be written as  $\mathcal{P}^\tau(E_2 | E_1; \mathcal{H}) \cdot \mathcal{P}^\tau(E_1; \mathcal{H})$  because at time  $\tau$  both  $E_1$  and  $E_2$  are uncertain events and one can contemplate the uncertainty about  $E_1$  supposing that  $E_2$  were to be true, or vice versa, contemplating  $E_2$  and  $E_1$ .

The calculus of probability does not interpret probability—that is, it does not tell us what probability means—nor is it concerned with issues such as the nature of uncertainty, whose uncertainty, whose history, how large should  $\mathcal{H}$  be, and most important, how to determine  $\mathcal{P}^\tau(E; \mathcal{H})$ , and how to make this number operational. The calculus simply provides us with a set of rules by which the uncertainties about two (or more) events combine or *cohere*. Any set of rules for combining uncertainties that are in violation of the rules given above are said to be

*incoherent* with respect to the calculus of probability. Below we discuss why these rules are necessary.

Several approaches have been proposed, some of which pay attention to the issue of “whose uncertainty?”, and some of which impose restrictions on what  $\mathcal{H}$  can and cannot contain. We list here some of these approaches that relate to the title “beyond probability” as: belief functions, possibility theory and fuzzy logic, upper and lower probabilities, Jeffrey's Rule of Combination, confidence limits, hypothesis testing with Type I and Type II errors, significance levels, maximum likelihood estimates, and goodness of fit tests. Some of these approaches have a normative foundation; others are *ad hoc*. In the next section, we will focus on probability and make a case for it.

## INTERPRETATIONS OF PROBABILITY

The interpretation, or the meaning of probability, and approaches for assigning initial probabilities has been the subject of much discussion and debate. Usually, this discussion is on assignment rather than on interpretation, but in this paper the focus is on the fundamental issue of interpretation.

According to Good (1965), there are about eleven ways of interpreting probability, four of which are prominent enough to describe. Since reliability is a probability, and since probability can be interpreted in several ways, it follows that there could also be several interpretations of reliability, and several ways of quantifying reliability. Some of these ways will permit quantification when hard data (e.g., as derived from experiments or tests) are not available, whereas the others require hard data for quantification. These different views lead to different treatments and understandings about decision making. Thus the issue of interpreting probability is very much germane to the circumstances under which reliability is assessed.

The four theories of probability are:

- The Classical Theory,
- The A Priori or Logical Theory,
- The Relative Frequency Theory, and
- The Personalistic or Subjective Theory.

Each of these theories subscribes to a particular interpretation of probability. The calculus is common to all of the four theories mentioned above. However, the assignment of initial probabilities (needed to make the calculus operational) depends on one's interpretation of probability. Thus to a user of probability, like a physicist, an engineer, a statistician, or a reliability (survival) analyst, the interpretation of probability is of great importance. For a discussion of the first two, see (Bement, et al., 2001); an overview of the key features of the last two theories follows.

## The Relative Frequency Theory

The origins of the relative frequency theory of probability can be traced back to Aristotle, though it is Venn who may have explicitly announced the idea in 1866. The mathematical development of this theory has been attributed to von Mises (1957) and its philosophical discourse is due to Reichenbach (1949).

Key ideas of this theory are:

- Probability is a measure of an empirical, objective and physical fact of the external world, independent of human attitudes, opinions, models and simulations. To von Mises, it is a part of a descriptive physical science; to Reichenbach it is a part of the theoretical structure of physics.
- Probability is never relative to evidence or opinion. Like mass, it is determined by observations on the nature of the real world.
- All probabilities can only be known *aposteriori*, i.e., only upon observation.

In the relative frequency theory, probability is a property of a *collective*, i.e., scenarios involving events that repeat again, and again, and again. Thus, it excludes from consideration one-of-a-kind and individual events, since such events do not possess a repetitive feature. Games of chance, like coin tossing, and social mass phenomena (like actuarial and insurance problems), are considered collectives.

A collective is a long sequence of observations for which there is sufficient reason to believe that the relative frequency of an observed attribute will tend to a limit if the observations are *indefinitely* continued. This limit is called the probability of the attribute within the collective.

The essential issue is that for the relative frequency theory to be invoked, we must first establish the existence of a collective, then establish the existence of limits, and we can then speak of the probability of encountering a certain attribute in the collective.

To von Mises, the role of probability theory is to derive probabilities from the old (initial probabilities) using the calculus of probability; the specification of initial probabilities is the job of a statistician. However, both von Mises and Reichenbach agreed that initial probabilities are to be obtained as relative frequencies. To von Mises, the equally likely values in dice games was a consequence of historical observations based on the word of mouth experience of generations. In actuarial applications, since a sizeable amount of data exists, von Mises requires that collective be identified, that a stable value of a relative frequency be identified over groups and over time, and that the stable value be used as the initial probability.

The most practical virtue of the relative frequency theory of probability is that it applies in cases where the indifference principle fails to hold (like the situation wherein the coins and dice are loaded). Its psychological virtue is that it claims to be objective and therefore scientific. Physical scientists are therefore attracted to it; probability is considered to be located in the objects themselves, as a property of the objects such as mass or volume, and not in our attitudes. Also, like mass and volume, probability can be discovered by the key tools of science, namely, experimentation, observation, and confirmation by experimental replication.

Criticisms of the theory stem from the fact that in order to invoke it, we first need to *i*) introduce a random collective, *ii*) define that probability is a random collective, *iii*) and specify that probability is a property of the collective and not an individual member of the collective. Collectives are difficult to construct in actual life. For example, tossing a coin an infinite number of times raises the question of how similar must the tosses be to be considered a collective. If they are identical, we will always observe the same outcome. If they are dissimilar, how much dissimilarity is allowed (if this can be assessed at all). Finally, relative frequency probability is never known, can never be known to exist (limits of sequences is an abstract mathematical notion), and its value can never be confirmed or disputed.

To interpret a number such as  $10^{-9}$  (for the failure rate) we must *i*) first conceptualize a collective (such as an infinite number of almost identical lunar probes), *ii*) focus on an attribute of this collective (say loss of navigation control), and *iii*) be prepared to accept the notion that probability is a property of this collective with respect to the attribute, and not any particular member (a probe) of this collective. The number  $10^{-9}$  reflects the feature of any individual encountering this attribute in the collective. Thus,  $10^{-9}$  is really a measure of encounter. This number can never be verified, nor can it be proved or disproved. It exists only as a mathematical limit.

Whereas collectives can be conceptualized with mass social phenomena (like actuarial tables, I.Q.s of individuals, etc.), and in topics of physics (such as the movement of gas particles) it is often difficult to do so in many other scenarios. Indeed, it was a sociologist, Quetelet, who introduced the idea of a collective. This notion was first embraced by physicists (who may have influenced von Mises), but was then rejected by individuals like Bohr and Schrodinger, in light of Heisenberg's "principle of uncertainty" which defined uncertainty and probability without the collective concept and closer to the subjectivist view described in the next section

Using the relative frequency view of probability, the  $\tau$  and  $\mathcal{H}$  have no role to play, so that  $\mathcal{P}(E; \mathcal{H}) = \mathcal{P}(E)$ . Similarly, expert testimonies, corporate memory, mathematical models and scientific information do not matter; only hard data on actual events goes into assessing the initial probabilities.

Thus to summarize, the heart of the relative frequency theory of probability lies in the notion of repetitive events (actual or conceived).

### **The Personalistic or Subjective Theory**

The personalistic or subjective theory was first proposed by Ramsey (1931), though Borel may have alluded to it in 1924. The theory was more fully developed by DeFinetti (1937), (1974) and by Savage (1954). The key idea of this theory is that there is no such a thing as an objective probability, and probability is a degree of belief of a given person at a given time. The degree of belief must be measured in some sense, and a person's degrees of belief must conform to each other in a certain way. The person in question is an idealized one, namely, one who behaves normatively.

The intensity of belief is difficult to quantify; thus, we must look at some property related to it. Ramsey and De Finetti both favored behavioristic approach where the degree of belief is expressed via a willingness to bet. Thus, the probability of an event is the amount, say  $p$ , that you are willing to bet, on a two-sided bet, in exchange for \$1, should the event occur. By a two-sided bet we mean staking  $(1-p)$  in exchange for \$1, should the event not occur.

The feature of *coherence* that is a part of this theory is the normative feature. It ensures that the degrees of belief do not conflict, i.e., the avoidance of a Dutch-Book or *heads I win, tails you lose*. Coherence is achieved by adhering to the calculus of probability.

The subjective theory of probability permits us to talk about the probability of a simple unique event or the probability of repetitive events. Because there is no notion of an absolute probability, the theory gives no guidance on how to obtain initial probabilities. Personal probabilists claim that objectivity in statistics is a fallacy, because model choice, the judgement of indifference (i.e., the notion of equipossible) the choice of  $p$ -values, and significance levels, etc., are all subjective. They also claim that it is impossible to give a satisfactory definition of the phrase *we know nothing before observing*. This phrase motivates a consideration of the relative frequency theory of probability.

Finally, under the personalistic theory, the number  $10^{-9}$  for the failure rate has an unambiguous interpretation. It means that the individual declaring such a number, based on all of  $\mathcal{H}$  (to include expert testimony, corporate memory, mathematical modeling, simulation, and hard data, if available) at time  $\tau$ , is prepared to stake  $\$10^{-9}$  in exchange of \$1, should a failure occur. Equivalently, that individual is also willing to stake  $\$(1-10^{-9})$  in exchange of \$1, if the event does not occur. The person is indifferent to either of these bets.

Because probability in the personalistic theory is one person's opinion, there is not such a thing as an unknown probability, or a correct probability, or an objective probability. A person's probabilities may be elicited by invoking the principle of indifference, or by a system of carefully conducted comparative wagers, or simply by asking. In this theory, any factor that an individual chooses to consider is relevant, and any coherent value is as good as another.

According to some, the fundamental principle of science is consistency, and inference based on facts (i.e., hard data). Personal probabilities do not stem from these features. Furthermore, declared probabilities may not reflect true belief, and it is literally impossible to ensure coherence in real situations that tend to be complex and not like games of chance. Another argument against this theory stems from the thesis that some individuals do not like to bet, especially when considering how the bet is selected, and thus may be reluctant to declare their probabilities. Also, the theory has no provision to ensure that individuals with identical background information will declare identical probabilities, and that given an individual's action, it is difficult to separate the individual's probabilities from his/her utilities.

Perhaps the most important argument against the theory of personal or subjective probability is that experiments by psychologists have shown that individuals do not declare probabilities that cohere, i.e., they do not act according to the dictates of the calculus of probability (Meyer and

Booker, 1991). A counter-argument to the above criticism is that the theory of personal probability is a normative one; it prescribes how we should act—not how we do act.

## BEYOND PROBABILITY

These two currently prominent theories described above have their strengths and criticisms. As mentioned in the introduction, modern reliability problems require inventive solutions and broad interpretations, making the subjective interpretation of probability more applicable for decision making under uncertainty and when hard data are sparse or impossible to obtain. That being said the issues surrounding these uncertainties require examination.

Several logic based approaches have been proposed to address uncertainty. Like probability theory, fuzzy set theory and fuzzy logic have a calculus or axiomatic base. Arguments continue over whether these axioms provide a coherence for fuzzy. To avoid those, we will opt for the viewpoint that they do according to the theory's founder, Zadeh (1965).

Probability theory (regardless of its interpretation) is based upon crisp set theory and adheres to the *law of the excluded middle*; that is, any outcome either belongs to a set or does not belong to a set. Fuzzy sets reject the law of the excluded middle. This seemingly simple difference in probability and fuzzy theories is the one of the major points of contention.

Fuzzy set theory can be considered a calculus for imprecision and is a mathematical construct in set theory that enhances classical set theory. In complex reliability problems (e.g., Meyer, Booker, and Bement, 1999) it is useful for quantification of a certain type of uncertainty—turning rules relating system condition to performance, into numeric functional representations of uncertainty.

For example, consider the set of integers

$$X = \{1, 2, \dots, 10\},$$

and define a subset, of  $X$ , where

$$A = \{x : x \in X \text{ and } x \text{ is "medium"}\}.$$

Defining  $A$  implies a precision in defining what is “medium”. However, there is a personalistic interpretation of “medium.” Most might agree that 5 is a “medium” integer. But what about 7? Is 7 “medium,” or is it “large”? We are uncertain about the classification of 7. Because of this vagueness, we are unable to define the subset in the classical or crisp set definition.

*Membership functions* are a way of dealing with the above vagueness (or uncertainty). Define  $\mu_A(x)$  as a membership function of  $A$  where  $\mu$  is (almost always, but not necessarily) a number between 0 and 1 that reflects the extent to which  $x \in A$ . Individuals (experts) assigns to each  $x \in X$  a number,  $\mu_A(x)$ , and this is done for all subsets of the type that are of interest. This set is called a *fuzzy set*. For crisp sets, all  $x \in X$ ,  $\mu_A(x) = 0$  or 1.

Continuing this example, we define a subset of  $X$ , where

$$B = \{x : x \in X \text{ and } x \text{ is "small"}\},$$

The membership functions for  $A$  and  $B$  are plotted in Figure 1.



In reliability applications, membership functions can be useful for transforming qualitative rules about how system conditions map into performance. For example, system conditions could be characterized by the set {good, nominal, poor} and represented by a three corresponding membership functions. Similarly, the performance could be represented by membership functions for the sets {normal, marginal, bad}. *If-then* rules map condition into performance (e.g., *if the condition is good then the performance is normal*), providing an uncertainty distribution of performance for each condition via the membership functions (Booker, et al. , 2000). This type of rule based uncertainty quantification is not directly achievable using probability.

However, research efforts are progressing to find common ground between the fuzzy and probability theories. (Ross, Booker, Parkinson, 2001, chapter 2). Specifically, it can shown that membership functions are *likelihoods*—the same likelihood functions found in the subjective-probability based Bayes Theorem. Bayes Theorem then becomes a natural bridge between fuzzy and probability. Also this bridge makes it possible for probability theory to have wider application in areas of imprecise uncertainty.

There may be other theoretical connections possible between probability and other logic paradigms such as possibility theory and belief functions. Such relationships would only strengthen the case for the flexibility of using personalistic or subjectivist probability.

## CONCLUSIONS

If probability is a way to quantify uncertainty, then it is also a way to quantify reliability. From a philosophical standpoint, the personalistic or subjectivist interpretation of probability does not lead to logical inconsistencies and the other difficulties of communication. Furthermore, it enables us to make statements of uncertainty about one-of-a-kind items, allows us to incorporate information from all sources deemed appropriate and does not demand the availability of a large amount of hard data nor preclude its use if available. It permits the incorporation of all knowledge we have at any given time with the ability to update our probabilities (and hence reliabilities) as new knowledge becomes available. A prime example where the formal use of all knowledge might have presented a different decision is the Challenger Space Shuttle tragedy. Instead of complete reliance on the solid rocket boosters hard data (from a relative frequency view), a personalistic approach may have revealed the potential problems which led to the disaster.

Another example of sparse hard data is the successful development and application of a reliability methodology, PREDICT (Meyer, Booker, and Bement, 1999). This methodology is a set of formal techniques to predict effectiveness and/or performance by mathematically combining all sources of data and information into an overarching process for decision making. Sources of data and information include formally elicited expert judgment, historical data and information, model outputs, simulations, and test data. Subjectivist probability is the unifying standard for estimating uncertainties associated with these sources and for combining them into an integrated reliability estimate (with uncertainty). Birth to death development of new auto

system designs (for Delphi Automotive Systems) and performance estimation of the aging nuclear physics package (for the Los Alamos Nuclear Weapons Program) are the two successful applications of these methods.

From a pragmatic point of view, the explosion of computational capabilities over the last few years has made knowledge and information available in a variety of forms, both qualitative and quantitative. Smart sensing networks coupled with computer simulation models can be used to diagnose and forecast system performance, providing predictions in advance of mechanical failures. However these sensors and tera-scale simulations (Farrar, Doebling and Nix, 2001) provide terabytes of information which must be analyzed and condensed for decision making. Such analyses rely on probability and involve inferences, again making the subjectivist interpretation the more appropriate choice.

Given these examples, we feel that subjectivist probability will continue to be the point of view that is most appropriate for addressing such dynamic and complex reliability problems. The subjectivist view of probability can provide such a paradigm for quantification of uncertainty and information/data integration for determining reliability which, in turn, is input into modern day decision making.

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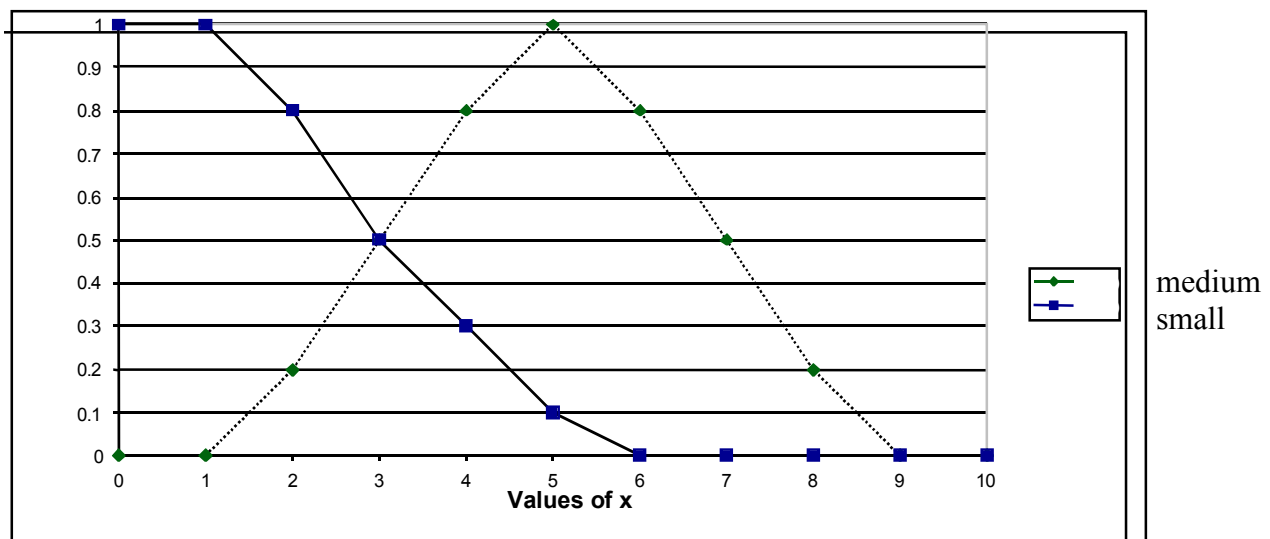


Figure 1. Membership functions for “medium” (dashed) and “small” (solid) integers.